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Surface defects in parallel-plate capacitors

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Abstract. It is shown that a parallel-plate capacitor with a sinusoidal defect in one of its electrodes has the same capacitance as a plane electrode capacitor with slightly smaller inter-electrode spacing, and the same force per unit area as a plane electrode capacitor with still smaller spacing. General expressions are given for the apparent reduction in spacing for both cases as a function of the wavelength and amplitude of the defect. Extensions of the theory to a general periodic defect, and to a capacitor with defects in both electrodes, are discussed.

1. Introduction

The parallel-plate capacitor is a paradigm of classical electrostatics, and the determination of its electric field strength, capacitance per unit area and other properties are trivial textbook exercises. Recently we have re-examined the departures from the simple theory which occur when the plates are of finite extent (Sloggett *et al* 1986). Here we consider theoretically another type of deviation from the ideal case, that due to surface imperfections in one or both plates. This investigation, like the earlier one, is motivated by the experimental development of a parallel-plate liquid electrometer for the absolute measurement of voltage (Clothier 1965, Sloggett *et al* 1985). In this instrument, precise measurements of the separation of the electrodes, one of which is a liquid mercury surface, are used to determine the electrostatic force existing between them. The electrode separation is variable in the range 2–5 mm, and is measured with a precision of about 0.1 nm. It is known that the mercury surface has waves of the order of 10 nm RMS amplitude, while the second electrode has polishing defects whose RMS amplitude may amount to a few nm. The question to be investigated is: to what extent do the surface irregularities modify the effective electrostatic spacing of the electrodes?

To simplify the problem we consider the electrode system shown in figure 1, in which one electrode is a conducting plane and the other has sinusoidal undulations of peak-to-peak amplitude $2a$ and wavelength L. Both electrodes are of infinite extent and their mean spacing is $d = 2y_0$. Clearly the electric field distribution will be neither vertical nor uniform close to the upper electrode. If the undulations are very deep ($a \gg L$), only a weak field will penetrate to the upper extrema of the sinusoidal electrode, while at its lower extrema the field strength will be enhanced. These field distortions accentuate the importance of the lower extrema in determining the effective electrostatic separation of the electrodes. Thus one might imagine that a plane electrode which is electrostatically equivalent to the sinusoidal electrode will lie not at its geometrical mean position $y = y_0$, but at a position intermediate between $y = y_0$ and $y = y_0 - a$.

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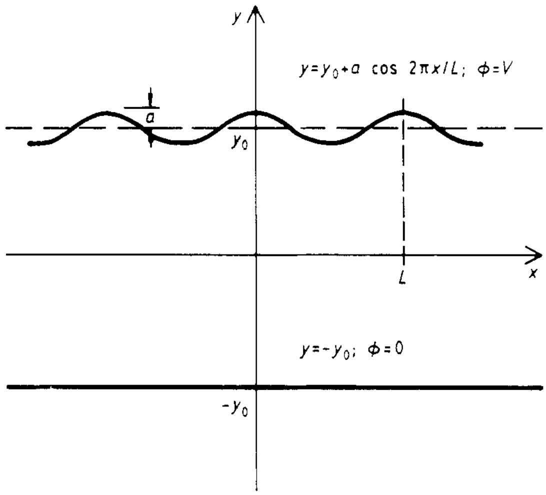

Figure 1. Electrode geometry.

A related problem, in which the periodic defect in the upper electrode has a different, non-sinusoidal form, has been studied by Maxwell (1904). For the electrode geometry studied he gives an expression for the inter-electrode spacing of an ideal parallel-plate capacitor whose capacitance is equal to that of the capacitor with defect. Below we give a similar result for the case of a sinusoidal defect, and in addition give an expression for the spacing of an ideal parallel-plate capacitor having the same inter-electrode force as the capacitor with sinusoidal defect. We first show that these are distinct criteria.

2. Equivalent plane electrodes

One sense in which a plane electrode could be said to be equivalent to the sinusoidal electrode of figure 1 would be that, when carrying the same potential V as the sinusoidal electrode, it produces the same mean electric field strength at the lower electrode. The inter-electrode field distribution $\phi(x, y)$ for figure 1 will be periodic in x with period L. Hence the mean field strength at $y=-y_0$ is

$$\bar{E}=-\frac{1}{L}\int_0^L\left(\frac{\partial\phi}{\partial y}\right)_{-y_0}\mathrm{d}x. \tag{1}$$

Let the plane of the flat electrode producing a uniform field of intensity $\bar{E}$ be $y=y_c$ and its distance from the lower electrode be d_c. Then

$$\frac{V}{d_c}=\frac{V}{y_0+y_c}=\frac{1}{L}\int_0^L\left(\frac{\partial\phi}{\partial y}\right)_{-y_0}\mathrm{d}x. \tag{2}$$

defines the position of this electrode.

Since the charge per unit area on the lower electrode is proportional to the field strength $(\partial\phi/\partial y)_{-y_0}$, the plane electrode pair $y=y_c$ and $y=-y_0$ will have the same mean capacitance per unit area as the system of figure 1. We may then refer to the electrode whose position is defined by equation (2) as the equal capacitance plane electrode, and to d_c as the electrode separation for equal capacitance. For a given value of the potential difference V, the equal capacitance plane electrode system will have the same stored energy (per unit area) as the sinusoidal electrode system.

A different criterion of equivalence is obtained if we consider the force of attraction between the electrodes. For the system of figure 1, with the voltage V held fixed, the mean force per unit area is

$$F = -\tfrac{1}{2} V^2 \frac{\partial C}{\partial d} \tag{3}$$

where C is the mean capacitance per unit area at the geometrical spacing d, given by

$$C = -\varepsilon \bar{E} / V. \tag{4}$$

Hence

$$F = \frac{\varepsilon V}{2} \frac{\partial \bar{E}(d)}{\partial d} \tag{5}$$

where $\bar{E}(d)$ is given by (1) with $y_0 = \tfrac{1}{2}d$. For an ideal parallel-plate capacitor of separation d_F, the force per unit area is

$$F = \frac{\varepsilon V^2}{2 d_F^2}. \tag{6}$$

Equating (5) and (6) we obtain

$$\frac{V}{d_F^2} = \frac{\partial \bar{E}(d)}{\partial d}$$

or, from (1),

$$d_F = \left\{ -\frac{1}{LV} \frac{\partial}{\partial d} \left[\int_0^L \left(\frac{\partial \phi}{\partial y} \right)_{-d/2} \mathrm{d}x \right] \right\}^{-1/2}. \tag{7}$$

Equation (7) gives the plate separation of a parallel-plate capacitor whose mean force per unit area is the same as that of figure 1 for the same applied voltage V. We may refer to d_F as the electrode separation for equal force. The plane of this electrode, which we may refer to as the equal force plane electrode, is $y = y_F$, where

$$y_F = d_F - y_0. \tag{8}$$

We thus have two distinct criteria for the equivalence of a parallel-plate electrode system and the sinusoidal plate system of figure 1: equal capacitance and equal force. In the following analysis both criteria are considered.

3. Analysis

We assume that the amplitude, a, is small compared to the spacing, d, and rewrite the profile of the upper electrode as

$$y = y_0(1 + \varepsilon \cos kx) \tag{9}$$

where $\varepsilon = a/y_0$ and $k = 2\pi/L$. We wish to solve Laplace's equation,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{10}$$

subject to the boundary conditions

$$\phi(x, -y_0) = 0 \tag{11a}$$

$$\phi(x, y_0(1+\varepsilon \cos kx)) = V. \tag{11b}$$

We assume a regular perturbation solution in powers of ε, namely

$$\phi(x, y, \varepsilon) = \phi_0(x, y) + \varepsilon\phi_1(x, y) + \varepsilon^2\phi_2(x, y) + \ldots \tag{12}$$

and treat the boundary condition (11*b*) by making a Taylor series expansion about $y = y_0$. This gives

$$\begin{aligned} V = \phi_0(x, y_0) &+ \varepsilon\left[\phi_1(x, y_0) + \left(\frac{\partial\phi_0}{\partial y}\right)_{y_0} y_0 \cos kx\right] \\ &+ \varepsilon^2\left[\phi_2(x, y_0) + \left(\frac{\partial\phi_1}{\partial y}\right)_{y_0} y_0 \cos kx + \frac{1}{2}\left(\frac{\partial^2\phi_0}{\partial y^2}\right)_{y_0} y_0^2 \cos^2 kx\right] + \ldots . \end{aligned} \tag{13}$$

It is therefore necessary to solve Laplace's equation for ϕ_0, ϕ_1, $\phi_2, \ldots$, using the boundary conditions

$$\phi_i(x, -y_0) = 0 \quad \text{for all } i \tag{14}$$

and, from (13),

$$\phi_0(x, y_0) = V \tag{15}$$

$$\phi_1(x, y_0) = -\left(\frac{\partial\phi_0}{\partial y}\right)_{y_0} y_0 \cos kx \tag{16}$$

$$\phi_2(x, y_0) = -\left(\frac{\partial\phi_1}{\partial y}\right)_{y_0} y_0 \cos kx - \frac{1}{2}\left(\frac{\partial^2\phi_0}{\partial y^2}\right)_{y_0} y_0^2 \cos^2 kx \tag{17}$$

and so on.

To second order in ε, we find that

$$\begin{aligned} \phi(x, y, \varepsilon) = \frac{V}{2y_0}(y + y_0) &+ \varepsilon\left(-\frac{V}{2}\frac{\sinh k(y+y_0)}{\sinh 2ky_0}\cos kx\right) \\ &+ \tfrac{1}{4}\varepsilon^2 V k y_0 \coth 2ky_0\left(\frac{y+y_0}{2y_0} + \frac{\sinh 2k(y+y_0)}{\sinh 4ky_0}\cos 2kx\right) + \mathrm{O}(\varepsilon^3). \end{aligned} \tag{18}$$

Neglecting terms in powers of ε greater than ε^2, equation (18) gives

$$\left(\frac{\partial\phi}{\partial y}\right)_{-y_0} = \frac{V}{2y_0} - \frac{\varepsilon V k \cos kx}{2\sinh 2ky_0} + \tfrac{1}{8}\varepsilon^2 V k \coth 2ky_0\left(1 + 4ky_0\frac{\cos 2kx}{\sinh 4ky_0}\right). \tag{19}$$

Substituting in (2) and integrating,

$$\frac{V}{d_c} = \frac{V}{2y_0}\left(1 + \frac{\varepsilon^2 y_0 k \coth 2ky_0}{4}\right). \tag{20}$$

Since the second term in (20) is much less than unity, we have

$$d_c = d\left(1 - \frac{\pi a^2 \coth 2\pi d/L}{dL}\right) \tag{21}$$

and so the equal capacitance plane electrode is displaced towards the lower electrode by the amount

$$\Delta d_c = d - d_c = \frac{\pi a^2}{L} \coth \frac{2\pi d}{L}. \tag{22}$$

To examine the equivalent force electrode, substitute (19) into (7) to obtain

$$\begin{aligned} d_F &= \left[-\frac{\partial}{\partial d} \left(\frac{1}{d} + \tfrac{1}{8}\varepsilon^2 k \coth kd \right) \right]^{-1/2} \\ &\approx d \left(1 - \frac{\pi^2 a^2}{L^2 \sinh^2 2\pi d/L} - \frac{\pi a^2}{Ld} \coth \frac{2\pi d}{L} \right). \end{aligned} \tag{23}$$

The displacement Δd_F of the equal force plane electrode towards the lower electrode is thus

$$\Delta d_F = d - d_F = \frac{\pi^2 a^2 d}{L^2 \sinh^2 2\pi d/L} + \frac{\pi a^2}{L} \coth \frac{2\pi d}{L}. \tag{24}$$

4. Discussion

Comparing (22) and (24), it is evident that Δd_F is always greater than Δd_c. It is of interest to consider the behaviour of Δd_c and Δd_F for two limiting cases, namely $2\pi d/L \gg 1$ (wavelength much less than the electrode separation) and $2\pi d/L \ll 1$ (wavelength much greater than the electrode separation). For $2\pi d/L \gg 1$, the first term in the RHS of (24) is negligible and Δd_c and Δd_F converge on a value which is independent of the plate separation:

$$\Delta d_F \approx \Delta d_c \approx \pi a^2/L. \tag{25}$$

For $2\pi d/L \ll 1$, Δd_F and Δd_c take different values, both of which are independent of the wavelength:

$$\Delta d_c \approx a^2/2d \tag{26}$$

$$\Delta d_F \approx 3a^2/4d. \tag{27}$$

Figure 2 shows representative curves computed from (22) and (24) for a range of wavelengths L and several amplitudes a. Curves are given for two spacings at the extremes of the range of interest in the liquid electrometer, $d = 2.0$ mm and $d = 4.6$ mm. The curves confirm the limiting behaviour discussed above.

Equations (22)–(27) are quadratic in a, indicating that the displacements Δd_c and Δd_F rapidly become negligible for small a. Both figure 2 and equation (25) indicate an inverse dependence on the wavelength L. Clearly Δd_c and Δd_F cannot increase beyond a, even for very large values of a or small values of L. There is a limit of validity for the analysis given here, arising from the neglect of higher-order terms in (18). We have found that the coefficient of ε^3 includes a term in $k^2 y_0^2$ which, to be negligible, requires $L \gg 2\pi a$, i.e. that the undulations be of small slope. This requirement, which supplements the other formal requirement of our derivation, $\varepsilon = 2a/d \ll 1$, excludes the region in which anomalously large displacements are predicted by (25).

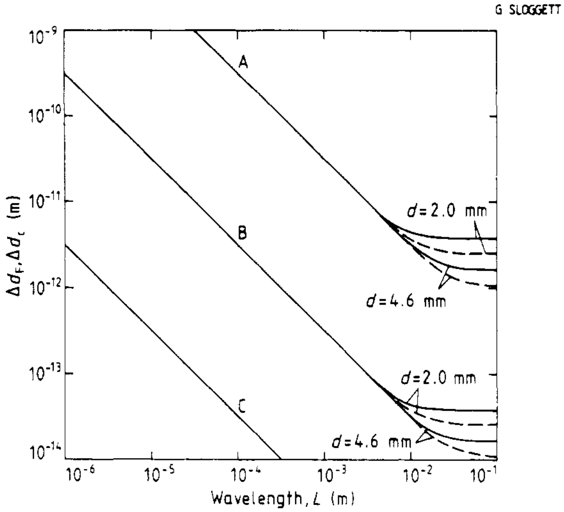

Figure 2. Displacements of equivalent plane electrodes as a function of the wavelength and amplitude of a sinusoidal electrode defect, for two electrode separations d. Full curve, Δd_F; broken curve, Δd_c. A, $a = 100$ nm; B, $a = 10$ nm; C, $a = 1$ nm.

A result of Maxwell (1904) is applicable to the case of deep undulations. He shows that, for a defect having a particular non-sinusoidal periodic form, in our notation,

$$\Delta d_c = a - \frac{L}{\pi} \ln \frac{2}{1 - e^{-2\pi a/L}} \tag{28}$$

where $2a$ is the peak-to-peak amplitude of the periodic functions considered. For very deep undulations, little electric field penetrates into the undulations and the electrostatic properties of the electrode are essentially determined by the form of the lower extrema of the undulations. In the limit $a/L \to \infty$ the form of the lower extrema of the upper electrode in both the sinusoidal case and the case treated by Maxwell approaches that of the set of semi-infinite planes $x = (m + \frac{1}{2})L$, $y > y_0 - a$, where $m = 0, \pm 1, \pm 2, \ldots$, etc. In this limit equation (28) gives a result which is therefore also applicable to the sinusoidal case:

$$\Delta d_c \to a - \frac{L \ln 2}{\pi}. \tag{29}$$

Thus, for deep undulations, Δd_c approaches a limit which is of the order of, but less than, a. Similar, if not identical, limiting behaviour is expected for Δd_F.

For the liquid electrometer it is relevant to know the extent to which the displacement of the equivalent plane electrodes varies with d. This may be examined by subtracting the displacements for the two values of d used in figure 2; the results are plotted in figure 3. It is interesting to note that the changes in Δd_c and Δd_F with d are largest for large wavelengths L, i.e. the range of L for which the displacements themselves are smallest. The rate of decrease of the change in displacement as the wavelength decreases is extremely fast. Thus, although figure 2 indicates that, for $a = 10$ nm, displacements can be expected to exceed 0.1 nm for $L < 0.3$ μm, they will be essentially independent of d and can be eliminated by differential measurements of the type employed in the liquid electrometer (Sloggett *et al* 1985).

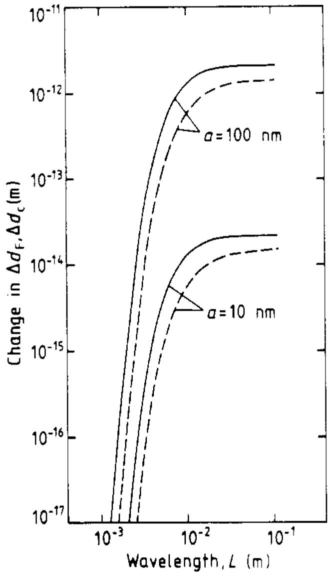

Figure 3. Change in displacements between the two electrode separations of figure 2. Full curve, change in Δd_F; broken curve, change in Δd_c.

The foregoing results may be extended to the case of an upper electrode which is an arbitrary periodic function of x with period L, namely

$$y = y_0 + \sum_{n=1}^{\infty} a_n \cos(2\pi nx/L + \theta_n). \tag{30}$$

By re-expressing this equation in the perturbation form

$$y = y_0\left(1 + \varepsilon \sum_{n=1}^{\infty} \alpha_n \cos(2\pi nx/L + \theta_n)\right) \tag{31}$$

and proceeding as for (9), it is found that the displacement of the equal capacitance plane electrode is

$$\Delta d_c = \sum_{n=1}^{\infty} \frac{\pi n a_n^2}{L} \coth \frac{2\pi nd}{L}. \tag{32}$$

The nth term of the above sum is just the displacement $\Delta d_c(n)$ due to the nth harmonic component of the surface roughness. Equation (32) therefore shows that the capacitance displacement due to a roughness profile composed of any number of harmonically related sinusoids is the linear sum of the displacements due to the individual sinusoids.

So far we have examined only the case of a single rough electrode opposed by a plane electrode. We conclude by considering briefly what might happen when both electrodes are rough. When the wavelength of the undulations is small compared with the electrode separation, the region of field non-uniformity is confined to the vicinity of the rough surface and does not extend to the opposite electrode. This is the case treated by equation (25) and accounts for its independence of the plate separation d. The displacements Δd_F and Δd_c may then be thought of as intrinsic properties of a rough electrode, independent of the position or properties of the second electrode.

The same will be true for displacements due to small-period undulations in the second electrode. The displacements will then be additive, so that the effective separation of a pair of rough electrodes of small wavelength will be

$$d_{\text{eff}} = d - \Delta d_1 - \Delta d_2 \tag{33}$$

where Δd_1 and Δd_2 are the displacements (in either the equal capacitance or equal force sense) due to surface roughness of the individual electrodes.

If L is comparable with, or greater than, d, the displacements associated with each electrode are influenced by the position, and possibly also the surface profile, of the other electrode. This is indicated by the dependence on d of (26) and (27). In this case it can no longer be assumed that displacements due to the individual surfaces are additive.

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